

# MODULI OF CONNECTIONS WITH A SMALL PARAMETER ON A CURVE

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**ABSTRACT.** We study  $GL_2$ -bundles with connections with a small parameter on a smooth projective curve. We describe an open subset in the moduli space of such bundles. The description degenerates into the Hitchin fibration as the parameter tends to zero.

## 1. INTRODUCTION

1.1. The moduli space of Higgs bundles on a curve admits a well-known description in terms of spectral curves (the Hitchin fibration). On the other hand, Higgs bundles can be viewed as a degeneration of bundles with connections: P. Deligne introduced the notion of ‘ $\lambda$ -connections’, and Higgs fields (resp. connections) are  $\lambda$ -connections for  $\lambda = 0$  (resp.  $\lambda = 1$ ). It is natural to ask whether spectral curves can be used to describe the moduli space of  $\lambda$ -connections for  $\lambda \neq 0$ .

The simplest case is when  $\lambda \in \mathbb{C}[[\lambda]]$  is a formal parameter; that is, the  $\lambda$ -connections considered are formal deformations of Higgs bundles. This case has the following advantage: if a  $\lambda$ -connection is a formal deformation of a Higgs bundle, we can try using the spectral curve corresponding to the Higgs bundle to describe the  $\lambda$ -connection. Informally, if  $\lambda$  is an actual number rather than a formal parameter (for instance,  $\lambda = 1$ ), we would not know which spectral curve to use.

Let  $\mathbf{Conn}_\lambda$  be the moduli space of  $\lambda$ -connections; more precisely, it parametrizes triples  $(L, \nabla, \lambda)$ , where  $L$  is a  $G$ -bundle on  $X$ ,  $\lambda \in \mathbb{C}$ , and  $\nabla$  is a  $\lambda$ -connection on  $L$ . Here  $X$  is a smooth curve and  $G$  is a reductive group. The moduli stack of Higgs bundles is the closed substack  $\mathbf{Higgs} \subset \mathbf{Conn}_\lambda$  given by the condition  $\lambda = 0$ . Making  $\lambda$  a formal parameter corresponds to working with the formal completion  $\mathbf{Conn}_{form}$  of  $\mathbf{Conn}_\lambda$  along  $\mathbf{Higgs}$  instead of  $\mathbf{Conn}_\lambda$  itself.

*Remark 1.1.* Although  $\lambda$ -connections are interesting geometric objects in their own right, they are particularly important because they can be used to compactify the moduli stack  $\mathbf{Conn}$  of ordinary connections ([6], [5]). We hope that studying  $\mathbf{Conn}_{form}$  can improve our understanding of  $\mathbf{Conn}$  (which is important, for instance, in the geometric Langlands program). One particular case ( $SL_2$ -bundles with connections on  $\mathbb{P}^1$  with four simple poles) appears in [1]: a statement about  $\mathbf{Conn}_{form}$  ([1, Proposition 6]) is used to compute the cohomology groups  $H^i(\mathbf{Conn}, F)$  for some natural coherent sheaves  $F$ .

The problem simplifies further if we consider formal deformations of only those Higgs bundles that are non-degenerate in some sense. Geometrically, this corresponds to taking an open substack  $\mathbf{Higgs}' \subset \mathbf{Higgs}$  (parametrizing non-degenerate

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bundles) and studying the formal completion  $\mathbf{Conn}'_{form}$  of  $\mathbf{Conn}_\lambda$  along **Higgs**. Let us look at three different non-degeneracy conditions for Higgs bundles.

Firstly, let us consider only the Higgs bundles with unramified spectral curves (equivalently, the Higgs field is regular semisimple). The answer in this case is relatively simple (see Theorem 3.1) and goes back to W. Wasow. However, from the geometric point of view, this non-degeneracy condition is too restrictive. Geometrically, the most interesting situation is when  $X$  is projective (so that **Higgs** and  $\mathbf{Conn}_\lambda$  are algebraic stacks); but if  $X$  is projective and not elliptic, it has no unramified spectral curves.

Secondly, let us allow ramifications, but only of ‘the simplest possible kind’. More precisely, if  $G = \mathrm{GL}_2$ , we consider Higgs bundles whose spectral curves are smooth (possibly ramified) covers of  $X$ . For arbitrary reductive  $G$ , roughly speaking, we consider Higgs bundles that can be locally on  $X$  reduced to Higgs bundles over  $\mathrm{GL}_2$  with smooth spectral curves (see Remark 1.2 for the precise condition). The main result of this paper describes  $\mathbf{Conn}'_{form}$  via spectral curves for this non-degeneracy condition.

*Remark 1.2.* For arbitrary reductive  $G$ , this non-degeneracy condition is most easily formulated using the notion of cameral covers. Recall ([2]) that a cameral cover is a cover of  $X$  that is locally isomorphic to the pull-back of the *universal cameral cover*  $\mathfrak{h} \rightarrow \mathfrak{h}/W$ , where  $\mathfrak{h}$  is the Cartan algebra of  $G$  and  $W$  is the Weyl group. To a Higgs bundle on  $X$ , there corresponds a cameral cover  $X_{cam} \rightarrow X$ ; locally on  $X$ , a Higgs field is essentially a map  $X \rightarrow \mathfrak{g}$  (where  $\mathfrak{g}$  is the Lie algebra of  $G$ ), and  $X_{cam} \rightarrow X$  is the pull-back of the universal cameral cover under the composition

$$X \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/G = \mathfrak{h}/W.$$

The second non-degeneracy condition on a Higgs bundle is that  $X_{cam}$  is smooth.

Finally, the last non-degeneracy condition allows a more general kind of ramifications. For instance, if  $G = \mathrm{GL}_m$ , the spectral curve is a degree  $m$  cover of  $X$ . Then let us work with Higgs bundles whose spectral curves are smooth. If  $m > 2$ , then this is a more general (and more natural) condition than the previous one, which only allows smooth curves whose ramification points have degree 2. It is possible to define this last non-degeneracy condition for an arbitrary reductive group, not just for  $\mathrm{GL}_m$ . However, to use this condition one needs to work with non-smooth cameral covers, which is more complicated. Moduli of  $\lambda$ -connections for this non-degeneracy conditions will be studied elsewhere.

In this paper, we work with the second non-degeneracy condition. We set  $G = \mathrm{GL}_2$  and consider Higgs bundles whose spectral curves are smooth (the results can then be extended to other reductive groups by using Levi subgroups). We use spectral curves to describe  $\lambda$ -connections that are formal deformations of such Higgs bundles (Theorem A), and then derive a description of  $\mathbf{Conn}'_{form}$  if  $X$  is projective (Theorem B).

**1.2. Conventions and notation.** In this work, the ground field is  $\mathbb{C}$ , that is, ‘scheme’ means ‘ $\mathbb{C}$ -scheme’,  $\mathrm{GL}_2$  means  $\mathrm{GL}_2(\mathbb{C})$ , and so on. However, our methods are purely algebraic, so our results hold over any algebraically closed field of characteristic zero.

For a scheme (or a formal scheme, or a stack)  $S$  and an integer  $m > 0$ , consider the following three versions of the category of  $\mathrm{GL}_m$ -bundles on  $S$ :

- the category  $\mathcal{Vect}$  of rank  $m$  vector bundles on  $S$ ;
- the subcategory  $\mathcal{Vect}^\times \subset \mathcal{Vect}$  of ‘invertible arrows’: the objects of  $\mathcal{Vect}^\times$  are rank  $m$  vector bundle of  $S$ , but arrows  $L_1 \rightarrow L_2$  are isomorphisms between  $L_1$  and  $L_2$ , rather than all homomorphisms ( $L_1, L_2 \in \mathcal{Vect}^\times$ );
- the category  $\mathcal{Princ}$  of principal  $\mathrm{GL}_m$ -bundles on  $S$ .

The categories  $\mathcal{Vect}^\times$  and  $\mathcal{Princ}$  are naturally equivalent.  $\mathcal{Vect}$  is a pre-additive category (morphisms between any two objects form a vector space), while  $\mathcal{Princ}$  and  $\mathcal{Vect}^\times$  are groupoids (all morphisms are invertible). Note also that  $\mathcal{Princ}$  makes sense for groups other than  $\mathrm{GL}_m$ , but  $\mathcal{Vect}$  does not.

In this paper, we work with  $\mathcal{Vect}^\times$ , which we call *the groupoid of  $\mathrm{GL}_m$ -bundles on  $S$*  (or **G<sub>m</sub>-bundles**, if  $m = 1$ ). The same convention applies to  $\mathrm{GL}_m$ -bundles with additional structures, such as connections,  $\lambda$ -connections, or Higgs fields. It is interesting to note however that Theorem A also holds in the ‘pre-additive’ settings; the proof is left to the reader.

## 2. MAIN RESULTS

**2.1.  $\lambda$ -connections on a curve.** Although our results hold for arbitrary reductive group, we prefer to formulate them for  $\mathrm{GL}_2$ . Let  $X$  be a smooth projective curve over  $\mathbb{C}$ .

**Definition 2.1.** Let  $L$  be a  $\mathrm{GL}_m$ -bundle on  $X$ ; that is,  $L$  is a locally free sheaf of rank  $m$ . A  $\lambda$ -connection on  $L$  (for some  $\lambda \in \mathbb{C}$ ) is a  $\mathbb{C}$ -linear map  $\nabla : L \rightarrow L \otimes \Omega_X$  which satisfies the  $\lambda$ -Leibniz identity:

$$(2.1) \quad \nabla(fs) = f\nabla(s) + \lambda s \otimes df$$

for any  $f \in \mathcal{O}_X$ ,  $s \in L$ .

*Example 2.2.* If  $\lambda = 1$ , we get the usual Leibniz identity, so  $\nabla$  is a connection on  $L$ . More generally, for any  $\lambda \neq 0$ ,  $\lambda^{-1}\nabla$  is a connection on  $L$ . On the other hand, if  $\lambda = 0$ ,  $\nabla$  is  $\mathcal{O}_X$ -linear (a *Higgs field*).

Denote by **Higgs** the moduli stack of *Higgs bundles*  $(L, \nabla)$  over  $\mathrm{GL}_2$  on  $X$ ; that is,  $L$  is a  $\mathrm{GL}_2$  bundle on  $X$  and  $\nabla$  is a  $\lambda$ -connection on  $L$  for  $\lambda = 0$ . The stack **Higgs** has a well-known geometric description (the Hitchin fibration) via spectral curves, which we remind in Theorem 2.4. Our aim is to provide a similar description for the moduli stack of  $\lambda$ -connections (or at least its open subset) when  $\lambda \in \mathbb{C}[[\lambda]]$  is a formal parameter.

Denote by **Conn <sub>$\lambda$</sub>**  the moduli stack of triples  $(L, \nabla, \lambda)$ , where  $\lambda \in \mathbb{C}$ ,  $L$  is a  $\mathrm{GL}_2$ -bundle on  $X$  and  $\nabla$  is a  $\lambda$ -connection on  $L$ . Then **Higgs** is identified with the closed substack of **Conn <sub>$\lambda$</sub>**  formed by triples  $(L, \nabla, \lambda)$  with  $\lambda = 0$ . We will describe an open subset in the formal completion of **Conn <sub>$\lambda$</sub>**  along **Higgs**.

Let us recall the geometric description of **Higgs** ([4], see also [2] for a much more general statement). Let  $p : T^*X \rightarrow X$  be the cotangent bundle.

**Definition 2.3.** A pure dimension 1 subscheme  $\tilde{X} \subset T^*X$  is a *spectral curve* (for  $\mathrm{GL}_2$ ) if the projection  $p_{\tilde{X}} := p|_{\tilde{X}} : \tilde{X} \rightarrow X$  is finite of degree 2.

Let  $\mu = \mu_{\tilde{X}} \in H^0(\tilde{X}, p_{\tilde{X}}^* \Omega_X)$  be the restriction of the natural 1-form  $\mu_{T^*X} \in H^0(T^*X, p^* \Omega_X)$ . Denote by **SCurv** the space of all spectral curves. **SCurv** is isomorphic to an affine space: the coordinates on **SCurv** are the coefficients of the equation for  $\tilde{X}$ .

**Theorem 2.4.** *Let  $(L, \nabla) \in \mathbf{Higgs}$  be a Higgs bundle.*

- (1) *There exists a unique spectral curve  $\tilde{X} \in \mathbf{SCurv}$  and a unique (up to a canonical isomorphism) coherent  $\mathcal{O}_{\tilde{X}}$ -module  $l$  such that  $L = (p_{\tilde{X}})_* l$  and  $\nabla = (p_{\tilde{X}})_* \mu$ . We call  $(\tilde{X}, l)$  the spectral data of  $(L, \nabla)$ .*
- (2) *If  $\tilde{X}$  is smooth,  $l$  is an invertible sheaf on  $\tilde{X}$ .*
- (3) *For a smooth spectral curve  $\tilde{X}$  and an invertible sheaf  $l$  on  $\tilde{X}$ , there is a unique (up to a canonical isomorphism)  $(L, \nabla) \in \mathbf{Higgs}$  such that  $(\tilde{X}, l)$  is the spectral data of  $(L, \nabla)$ .*

□

*Remark 2.5.* Consider the morphism  $p_H : \mathbf{Higgs} \rightarrow \mathbf{SCurv}$  that sends a Higgs bundle to its spectral curve (the Hitchin fibration). Then Theorem 2.4 implies that the fiber of  $p_H$  over a smooth spectral curve  $\tilde{X} \in \mathbf{SCurv}$  is the moduli stack  $\mathbf{Pic}(\tilde{X})$  of line bundles on  $\tilde{X}$ .

Our first result is a version of Theorem 2.4 for  $\lambda$ -connections. Let us start with some definitions.

**Definition 2.6.** A  $\mathbb{C}[[\lambda]]$ -family of  $\mathrm{GL}_m$ -bundles with  $\lambda$ -connections on a smooth curve  $X$  is a pair  $(L, \nabla)$ , where  $L$  is a  $\mathrm{GL}_m$ -bundle on the formal scheme  $X[[\lambda]] := \varprojlim X \times \mathrm{Spec} \mathbb{C}[\lambda]/(\lambda^i)$ , and  $\nabla : L \rightarrow L \otimes_{\mathcal{O}_X} \Omega_X$  is a  $\mathbb{C}[[\lambda]]$ -linear  $\lambda$ -connection.

The *reduction* of  $(L, \nabla)$  modulo  $\lambda$  is the Higgs bundle  $(L_0, \nabla_0)$  on  $X$ , where  $L_0 := L/\lambda L$  is the restriction of  $L$  to  $X$ , and  $\nabla_0 : L_0 \rightarrow L_0 \otimes \Omega_X$  is induced by  $\nabla$ .

Let us fix a smooth (but not necessarily projective) curve  $X$  and a smooth spectral curve  $\tilde{X} \subset T^*X$ . Denote by  $\mathrm{Conn}_\lambda(\tilde{X})$  the groupoid of  $\mathbb{C}[[\lambda]]$ -families  $(L, \nabla)$  of  $\mathrm{GL}_2$ -bundles with connections on  $X$  such that  $\tilde{X}$  equals the spectral curve of  $(L_0, \nabla_0)$  (where  $(L_0, \nabla_0)$  is the reduction of  $(L, \nabla)$  modulo  $\lambda$ ). We would like to describe  $\mathrm{Conn}_\lambda(\tilde{X})$  in terms of bundles on the spectral curve  $\tilde{X}$ .

Denote by  $\widetilde{\mathrm{Conn}}_\lambda(\tilde{X})$  the groupoid of  $\mathbb{C}[[\lambda]]$ -families  $(l, \delta)$  of  $\mathbf{G}_m$ -bundles with connections on  $\tilde{X}$  such that  $\delta : l \rightarrow l \otimes \Omega_{\tilde{X}}(\tilde{x}_1 + \cdots + \tilde{x}_n)$  has first order poles at  $\tilde{x}_1, \dots, \tilde{x}_n$  (the ramification locus of  $p_{\tilde{X}} : \tilde{X} \rightarrow X$ ), the residue of  $\delta$  at  $\tilde{x}_i$  equals  $-\lambda/2$  (the notion of residue of a  $\lambda$ -connection is straightforward), and the reduction  $\delta_0$  of  $\delta$  modulo  $\lambda$  equals  $\mu \in H^0(\tilde{X}, \Omega_{\tilde{X}})$ . Notice that  $\delta_0$  is a Higgs field on the line bundle  $l/\lambda l$ , and a Higgs field on a line bundle is just a differential form.

**Theorem A.** *There exists an equivalence of categories  $\mathcal{F} : \mathrm{Conn}_\lambda(\tilde{X}) \xrightarrow{\sim} \widetilde{\mathrm{Conn}}_\lambda(\tilde{X})$  for a smooth (not necessarily projective) curve  $X$  and a smooth spectral curve  $\tilde{X} \subset T^*X$ .*

*Remark.* Theorem A is significantly simplified if  $\tilde{X}$  is unramified over  $X$  (see Theorem 3.1). This special case goes back to Wasow ([7, Theorem 25.2]).

*Remark 2.7.* The groupoid  $\widetilde{\mathrm{Conn}}_\lambda(\tilde{X})$  has a simpler description. Namely, for any  $(l, \delta) \in \widetilde{\mathrm{Conn}}_\lambda(\tilde{X})$ , the formula  $\partial := \lambda^{-1}(\delta - \mu)$  defines a connection  $\partial : l \rightarrow l \otimes \Omega_{\tilde{X}}(\tilde{x}_1 + \cdots + \tilde{x}_n)$ . In this manner,  $\widetilde{\mathrm{Conn}}_\lambda(\tilde{X})$  identifies with the groupoid of pairs  $(l, \partial)$ , where  $l$  is a line bundle on  $\tilde{X}[[\lambda]]$ , and  $\partial : l \rightarrow l \otimes \Omega_{\tilde{X}}(\tilde{x}_1 + \cdots + \tilde{x}_n)$  is a  $(\mathbb{C}[[\lambda]])$ -linear connection whose residues at  $\tilde{x}_i \in \tilde{X}$  equal  $-1/2$ .

The formulation of Theorem A is somewhat unsatisfactory, because the equivalence  $\mathcal{F}$  is not described. However, there are some natural compatibility conditions on  $\mathcal{F}$ . For instance, if  $X' \subset X$  is an open set,  $\tilde{X}' := X' \times_X \tilde{X}$  is a spectral curve over  $X'$ , and it is natural to ask that  $\mathcal{F}$  commutes with the restriction functors  $\text{Conn}_\lambda(\tilde{X}) \rightarrow \text{Conn}_\lambda(\tilde{X}')$ ,  $\widetilde{\text{Conn}}_\lambda(\tilde{X}) \rightarrow \widetilde{\text{Conn}}_\lambda(\tilde{X}')$ ; essentially, this corresponds to viewing  $\text{Conn}_\lambda(\tilde{X})$ ,  $\widetilde{\text{Conn}}_\lambda(\tilde{X})$  as stacks in the Zariski topology (the étale topology also works). Also, one naturally wants  $\mathcal{F}$  to be compatible with Theorem 2.4: for  $(L, \nabla) \in \text{Conn}_\lambda(\tilde{X})$ , the spectral data of its reduction  $(L_0, \nabla_0)$  should be canonically isomorphic to  $(\tilde{X}, l/\lambda l)$ , where  $(l, \delta) = \mathcal{F}(L, \nabla)$ . In some sense, the compatibility conditions determine  $\mathcal{F}$  up to a unique isomorphism, see Theorem 3.2.

**2.2. Moduli space of  $\lambda$ -connections.** We saw that Theorem 2.4 provides a geometric description of an open substack of **Higgs** (Remark 2.5). Similarly, Theorem A can be used to describe an open stack in the completion of **Conn** $_\lambda$  along **Higgs**.

Let  $X$  be a smooth projective curve. Denote by  $\mathbf{M}_\sharp$  the moduli stack of collections  $(\tilde{X}, l, \partial)$ , where  $\tilde{X} \in \text{SCurv}$  is a smooth spectral curve,  $l$  is a line bundle on  $\tilde{X}$ , and  $\partial : l \rightarrow l \otimes \Omega_{\tilde{X}}(\tilde{x}_1 + \cdots + \tilde{x}_n)$  is a connection (not a  $\lambda$ -connection) whose residues at  $\tilde{x}_1, \dots, \tilde{x}_n$  equal  $-1/2$ . As before,  $\tilde{x}_1, \dots, \tilde{x}_n$  are the ramification points of  $p_{\tilde{X}} : \tilde{X} \rightarrow X$ .

Consider the projection

$$p_\sharp : \mathbf{M}_\sharp \rightarrow \mathbf{Higgs} : (\tilde{X}, l, \delta) \mapsto (\tilde{X}, l);$$

here we use Theorem 2.4 to identify Higgs bundles with their spectral data  $(\tilde{X}, l)$ . The fiber of  $p_\sharp$  over  $(\tilde{X}, l)$  is the space of connections  $\partial : l \rightarrow l \otimes \Omega_{\tilde{X}}(\tilde{x}_1 + \cdots + \tilde{x}_n)$ ,  $\text{res}_{\tilde{x}_i} \partial = -1/2$ . The following statement is immediate:

**Lemma 2.8.** *Denote by  $\mathbf{Higgs}' \subset \mathbf{Higgs}$  the open substack of Higgs bundles whose spectral data  $(\tilde{X}, l)$  satisfy two conditions:  $\tilde{X}$  is smooth, and  $\deg(l) = n/2 = 2g - 2$ , where  $n$  is the number of ramification points of  $p_{\tilde{X}}$  and  $g$  is the genus of  $X$ . Equivalently,  $(L, \nabla) \in \mathbf{Higgs}'$  if its spectral curve is smooth and  $\deg(L) = 0$ . Then*

- (1)  $p_\sharp(\mathbf{M}_\sharp) = \mathbf{Higgs}'$ .
- (2) *For  $(\tilde{X}, l) \in \mathbf{Higgs}'$ , the fiber  $p_2^{-1}(\tilde{X}, l)$  is an affine space; the corresponding vector space is  $H^0(\tilde{X}, \Omega_{\tilde{X}})$ . More precisely: as  $\tilde{X}$  varies, the spaces  $H^0(\tilde{X}, \Omega_{\tilde{X}})$  form a vector bundle on  $\mathbf{Higgs}'$ , and  $p_2 : \mathbf{M}_\sharp \rightarrow \mathbf{Higgs}'$  is a torsor over this vector bundle.*

□

Denote by  $\zeta_0$  the relative tangent bundle to  $p_\sharp$ ; it is a foliation on  $\mathbf{M}_\sharp$ , and  $\mathbf{Higgs}'$  can be viewed as the quotient of  $\mathbf{M}_\sharp$  modulo  $\zeta_0$ .

*Remark 2.9.* Technically,  $\mathbf{M}_\sharp$  is an algebraic stack rather than a scheme, and the notion of a foliation on a stack requires clarification. However, the stack structure on  $\mathbf{M}_\sharp$  (and on  $\mathbf{Higgs}'$ ) is rather simple: the automorphism group of every point equals  $\mathbf{G}_m$ ; that is,  $\mathbf{M}_\sharp$  is a  $\mathbf{G}_m$ -gerbe over the corresponding coarse moduli space,  $M_\sharp$ . If we choose to work with  $M_\sharp$  instead of  $\mathbf{M}_\sharp$ , then  $\zeta_0$  becomes just a foliation on a smooth algebraic space; the downside is that in this way we get a description of the coarse moduli space of  $\lambda$ -connections rather than the true moduli stack. We

could avoid this difficulty if we rigidify the moduli problem, for instance, by adding a framing of vector bundles at some points.

On the other hand, it is not hard to define the notion of a foliation on an algebraic stack (for instance, using Lee algebroids). From now on, we will ignore this difficulty and freely use foliations on  $\mathbf{M}_\sharp$ .

Notice that  $\mathbf{M}_\sharp$  carries another foliation, which is defined via isomonodromic deformation. Let us consider the composition  $p_H \circ p_\sharp : \mathbf{M}_\sharp \rightarrow \text{SCurv}$ . The fiber of  $p_H \circ p_\sharp$  over a smooth spectral curve  $\tilde{X} \in \text{SCurv}$  is canonically identified with fibers over infinitesimally close spectral curves (the fiber is essentially the space of rank 1 local systems on  $\tilde{X}$  with monodromy  $-1$  around the ramification points; therefore, the fiber does not change under deformations of  $\tilde{X}$ ). More precisely, the morphism  $p_H \circ p_\sharp : \mathbf{M}_\sharp \rightarrow \text{SCurv}$  carries a connection. Let  $\zeta_\infty$  be the foliation (on  $\mathbf{M}_\sharp$ ) of horizontal vector fields with respect to this connection.

Let us now consider  $\zeta_0$  and  $\zeta_\infty$  as abstract vector bundles (rather than foliations) on  $\mathbf{M}_\sharp$ . Over a point  $(\tilde{X}, l, \partial) \in \mathbf{M}_\sharp$ , the fiber of  $\zeta_0$  equals  $H^0(\tilde{X}, \Omega_{\tilde{X}})$ , while the fiber of  $\zeta_\infty$  equals  $H^0(\tilde{X}, N_{\tilde{X}})$ , where  $N_{\tilde{X}}$  is the normal bundle to  $\tilde{X} \subset T^*X$ . The symplectic structure on  $T^*X$  identifies  $N_{\tilde{X}}$  with  $\Omega_{\tilde{X}}$ ; therefore,  $\zeta_0$  and  $\zeta_\infty$  are isomorphic as vector bundles on  $\mathbf{M}_\sharp$ .

*Remark 2.10.* For the isomorphism  $\Omega_{\tilde{X}} \xrightarrow{\sim} N_{\tilde{X}}$ , there are two choices that differ by sign; we choose the sign so that the diagram

$$\begin{array}{ccc} p_{\tilde{X}}^* \Omega_X & \rightarrow & T(T^*X)|_{\tilde{X}} \\ \downarrow & \xrightarrow{\sim} & \downarrow \\ \Omega_{\tilde{X}} & & N_{\tilde{X}} \end{array}$$

commutes. Here  $T(T^*X)|_{\tilde{X}}$  is the restriction to  $\tilde{X} \subset T^*X$  of the tangent bundle to  $T^*X$ , the map  $p_{\tilde{X}}^* \Omega_X \rightarrow T(T^*X)|_{\tilde{X}}$  identifies  $p_{\tilde{X}}^* \Omega$  with the subbundle of vertical vector fields,  $p_{\tilde{X}}^* \Omega_X \rightarrow \Omega_{\tilde{X}}$  is the pull-back map for differential forms, and  $T(T^*X)|_{\tilde{X}} \rightarrow N_{\tilde{X}}$  is the natural projection.

**Definition 2.11.** Let  $\zeta_0, \zeta_\infty \subset TM$  be distributions on a smooth variety  $M$ , and let  $\nu : \zeta_0 \xrightarrow{\sim} \zeta_\infty$  be an isomorphism of vector bundles on  $M$ . The *linear combination*  $\alpha\zeta_0 + \beta\zeta_\infty \subset TM$  is the distribution on  $M$  that is the image of the morphism  $\alpha(\text{id}_{\zeta_0}) + \beta\nu : \zeta_0 \rightarrow TM$ , provided the morphism is an embedding of vector bundles. Clearly, the linear combination  $\alpha\zeta_0 + \beta\zeta_\infty$ , if it exists, depends only on the ratio  $(\alpha : \beta) \in \mathbb{P}^1$ .

Notice that a linear combination is not necessarily a foliation even if  $\zeta_0$  and  $\zeta_\infty$  are foliations.

**Theorem B.** Let  $\mathbf{M}_\sharp$ ,  $\zeta_0$ , and  $\zeta_\infty$  be as above, and let us use the isomorphism  $\zeta_0 \xrightarrow{\sim} \zeta_\infty$  from Remark 2.10 to construct the linear combination  $\zeta_\lambda := \zeta_0 - \lambda\zeta_\infty$ ,  $\lambda \in \mathbb{C}$ .

- (1)  $\zeta_\lambda = \zeta_0 - \lambda\zeta_\infty$  is a foliation on  $\mathbf{M}_\sharp$  for any  $\lambda \in \mathbb{C}$ .
- (2) The quotient  $\mathbf{M}_\sharp/\zeta_\lambda$  exists if  $\lambda \in \mathbb{C}[[\lambda]]$  is a formal parameter, and such quotients form a family  $\mathbf{M}_\sharp[[\lambda]]/\zeta_\lambda \rightarrow \text{Spf } \mathbb{C}[[\lambda]]$  over the formal disc.
- (3)  $\mathbf{M}_\sharp[[\lambda]]/\zeta_\lambda$  is canonically isomorphic to the formal completion of  $\mathbf{Conn}_\lambda$  along **Higgs'**. This isomorphism respects the projection to  $\text{Spf } \mathbb{C}[[\lambda]]$  (intuitively,  $\mathbf{M}_\sharp/\zeta_\lambda$  is identified with an open substack in the moduli stack of  $\lambda$ -connections when  $\lambda$  is a formal parameter).

*Remark 2.12.* Let us show that the linear combination  $\zeta_\lambda$  exists (as a distribution) for any  $\lambda \in \mathbb{C}$ . Indeed, if  $\lambda = 0$ , then  $\zeta_\lambda = \zeta_0$ ; so it is enough to analyze the case  $\lambda \neq 0$ . Now consider the projection  $p_H \circ p_\# : \mathbf{M}_\# \rightarrow \text{SCurv}$ . Its differential  $d(p_H \circ p_\#)$  vanishes on  $\zeta_0$  and induces an isomorphism between  $\zeta_\infty \subset T\mathbf{M}_\#$  and the pull-back of the tangent bundle from  $\text{SCurv}$  to  $\mathbf{M}_\#$ . Therefore,  $d(p_H \circ p_\#)$  also induces an isomorphism between  $\zeta_\lambda$  and this pull-back. This implies the statement.

*Remark 2.13.* Note that although  $\mathbf{Higgs}'$  is open in  $\mathbf{Higgs}$ , it is not dense. Actually,  $\mathbf{Higgs}$  is disconnected; its connected components are

$$\mathbf{Higgs}^{(k)} := \{(L, \nabla) \in \mathbf{Higgs} \mid \deg(L) = k\} \quad (k \in \mathbb{Z}),$$

and  $\mathbf{Higgs}' \subset \mathbf{Higgs}^{(0)}$ . However, only the neighborhood of  $\mathbf{Higgs}^{(0)} \subset \mathbf{Conn}_\lambda$  is interesting, because  $\mathbf{Higgs}^{(k)} \subset \mathbf{Conn}_\lambda$  is a connected component of  $\mathbf{Conn}_\lambda$  for  $k \neq 0$ . This is easy to see by using the exterior square (the ‘trace’) of  $\lambda$ -connections.

**2.3. Organization.** In Section 3, we formulate a more precise version of Theorem A. We then show that the theorem follows from its ‘formal’ version (Theorem 3.4), in which  $X$  is a formal disc rather than a curve. Theorem 3.4 is proved in Section 4. Finally, we prove Theorem B in Section 5.2.

### 3. REDUCTION OF THEOREM A TO FORMAL DISC

**3.1. Refinement of Theorem A.** Let us now make Theorem A more precise. As before,  $X$  is a smooth curve,  $\tilde{X} \subset T^*X$  is a smooth spectral curve,  $p_{\tilde{X}} : \tilde{X} \rightarrow X$  is the projection,  $\{\tilde{x}_1, \dots, \tilde{x}_n\} \subset \tilde{X}$  is the ramification locus of  $p_{\tilde{X}}$ . Set  $\tilde{X}_u := \tilde{X} - \{\tilde{x}_1, \dots, \tilde{x}_n\}$ ,  $X_u := p_{\tilde{X}}(\tilde{X}_u) \subset X$ . We then have the following commutative diagram of groupoids:

$$(3.1) \quad \begin{array}{ccc} \text{Conn}_\lambda(\tilde{X}) & \rightarrow & \text{Conn}_\lambda(\tilde{X}_u) \\ \downarrow & & \downarrow \\ \mathcal{Higgs}(\tilde{X}) & \rightarrow & \mathcal{Higgs}(\tilde{X}_u), \end{array}$$

where  $\mathcal{Higgs}(\tilde{X})$  (resp.  $\mathcal{Higgs}(\tilde{X}_u)$ ) is the groupoid of Higgs bundles on  $X$  (resp.  $X_u$ ) whose spectral curve is  $\tilde{X}$  (resp.  $\tilde{X}_u$ ), and  $\text{Conn}_\lambda(\tilde{X})$  (resp.  $\text{Conn}_\lambda(\tilde{X}_u)$ ) is the groupoid of  $\mathbb{C}[[\lambda]]$ -families of  $\lambda$ -connections on  $X$  (resp.  $X_u$ ) from Theorem A. In the diagram (3.1), the horizontal arrows are functors of restriction from  $X$  to  $X_u$  and the vertical arrows are functors of reduction modulo  $\lambda$ .

Similarly, the groupoid  $\widetilde{\text{Conn}}_\lambda(\tilde{X})$  fits into the commutative diagram

$$(3.2) \quad \begin{array}{ccc} \widetilde{\text{Conn}}_\lambda(\tilde{X}) & \rightarrow & \widetilde{\text{Conn}}_\lambda(\tilde{X}_u) \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{Higgs}}(\tilde{X}) & \rightarrow & \widetilde{\mathcal{Higgs}}(\tilde{X}_u). \end{array}$$

Here  $\widetilde{\mathcal{Higgs}}(\tilde{X})$  (resp.  $\widetilde{\mathcal{Higgs}}(\tilde{X}_u)$ ) is the groupoid of line bundles on  $\tilde{X}$  (resp.  $\tilde{X}_u$ ).

Theorem 2.4 provides an equivalence  $\mathcal{Higgs}(\tilde{X}) \xrightarrow{\sim} \widetilde{\mathcal{Higgs}}(\tilde{X})$  and an equivalence  $\mathcal{Higgs}(\tilde{X}_u) \xrightarrow{\sim} \widetilde{\mathcal{Higgs}}(\tilde{X}_u)$ . So we see that the bottom rows of diagrams (3.1), (3.2) are naturally equivalent. By the following statement, their upper right corners are also equivalent:

**Theorem 3.1.** *Suppose  $X$  is a smooth curve and  $p_{\tilde{X}} : \tilde{X} \rightarrow X$  is an unramified spectral curve. The functor  $\widetilde{\text{Conn}}_{\lambda}(\tilde{X}) \rightarrow \text{Conn}_{\lambda}(\tilde{X})$  that sends  $(l, \delta)$  to  $((p_{\tilde{X}})_*(l), (p_{\tilde{X}})_*(\delta))$  is an equivalence of groupoids.  $\square$*

This theorem is a bit generalized version of [3, Proposition 1.2] (see also [7, Theorem 25.2]) and can be proved by the same method. We are now ready to refine the statement of Theorem A:

**Theorem 3.2.** *Let  $X$  be a smooth curve and  $\tilde{X} \subset T^*X$  a smooth spectral curve. Consider the fibered products of groupoids:*

$$(3.3) \quad \mathcal{C}(\tilde{X}) := \text{Conn}_{\lambda}(\tilde{X}_u) \times_{\mathcal{H}iggs(\tilde{X}_u)} \mathcal{H}iggs(\tilde{X})$$

$$(3.4) \quad \tilde{\mathcal{C}}(\tilde{X}) := \widetilde{\text{Conn}}_{\lambda}(\tilde{X}_u) \times_{\widetilde{\mathcal{H}iggs}(\tilde{X}_u)} \widetilde{\mathcal{H}iggs}(\tilde{X}).$$

Notice that Theorems 2.4 and 3.1 give an equivalence  $\mathcal{C}(\tilde{X}) \xrightarrow{\sim} \tilde{\mathcal{C}}(\tilde{X})$ .

- (1) The functor  $\text{Conn}_{\lambda}(\tilde{X}) \rightarrow \mathcal{C}(\tilde{X})$  induced by (3.1) is fully faithful (so that  $\text{Conn}_{\lambda}(\tilde{X})$  is a full subcategory of  $\mathcal{C}(\tilde{X})$ ).
- (2) The functor  $\widetilde{\text{Conn}}_{\lambda}(\tilde{X}) \rightarrow \tilde{\mathcal{C}}(\tilde{X})$  induced by (3.2) is fully faithful.
- (3) For a groupoid  $\mathcal{G}$ , let  $[\mathcal{G}]$  be the set of isomorphism classes of objects of  $\mathcal{G}$ . The equivalence  $\mathcal{C}(\tilde{X}) \xrightarrow{\sim} \tilde{\mathcal{C}}(\tilde{X})$  induces an isomorphism  $[\mathcal{C}(\tilde{X})] \xrightarrow{\sim} [\tilde{\mathcal{C}}(\tilde{X})]$ . We claim that this isomorphism identifies the sets  $[\text{Conn}_{\lambda}(\tilde{X})] \subset [\mathcal{C}(\tilde{X})]$  and  $[\widetilde{\text{Conn}}_{\lambda}(\tilde{X})] \subset [\tilde{\mathcal{C}}(\tilde{X})]$ .

**Remark 3.3.** It is obvious that the restriction functors  $\text{Conn}_{\lambda}(\tilde{X}) \rightarrow \text{Conn}_{\lambda}(\tilde{X}_u)$  and  $\widetilde{\text{Conn}}_{\lambda}(\tilde{X}) \rightarrow \widetilde{\text{Conn}}_{\lambda}(\tilde{X}_u)$  are faithful. Therefore, the functors  $\text{Conn}_{\lambda}(\tilde{X}) \rightarrow \mathcal{C}(\tilde{X})$  and  $\widetilde{\text{Conn}}_{\lambda}(\tilde{X}) \rightarrow \tilde{\mathcal{C}}(\tilde{X})$  are also automatically faithful.

Theorem 3.2 claims that the equivalence  $\mathcal{C}(\tilde{X}) \xrightarrow{\sim} \tilde{\mathcal{C}}(\tilde{X})$  induces an equivalence  $\text{Conn}_{\lambda}(\tilde{X}) \xrightarrow{\sim} \widetilde{\text{Conn}}_{\lambda}(\tilde{X})$  that is unique up to a canonical isomorphism. Therefore, Theorem 3.2 implies Theorem A.

3.2. It is easy to see that all of the above definitions ( $\lambda$ -connections, Higgs bundles, spectral curves, etc.) still make sense if  $X$  is a formal disc rather than a smooth curve (see Section 3.3 for examples). Therefore, we can formulate a ‘formal’ version of Theorem 3.2:

**Theorem 3.4.** *Let  $X \simeq \text{Spf } \mathbb{C}[[z]]$  be a formal disc and  $\tilde{X} \subset T^*X$  a smooth spectral curve. Define  $\mathcal{C}(\tilde{X})$  and  $\tilde{\mathcal{C}}(\tilde{X})$  by (3.3), (3.4).*

- (1) The natural functor  $\text{Conn}_{\lambda}(\tilde{X}) \rightarrow \mathcal{C}(\tilde{X})$  is fully faithful.
- (2) The natural functor  $\widetilde{\text{Conn}}_{\lambda}(\tilde{X}) \rightarrow \tilde{\mathcal{C}}(\tilde{X})$  is fully faithful.
- (3) Let us identify  $[\mathcal{C}(\tilde{X})]$  and  $[\tilde{\mathcal{C}}(\tilde{X})]$  using the equivalence  $\mathcal{C}(\tilde{X}) \xrightarrow{\sim} \tilde{\mathcal{C}}(\tilde{X})$ . We claim that under this identification,  $[\text{Conn}_{\lambda}(\tilde{X})] \subset [\mathcal{C}(\tilde{X})]$  corresponds to  $[\widetilde{\text{Conn}}_{\lambda}(\tilde{X})] \subset [\tilde{\mathcal{C}}(\tilde{X})]$ .

We will prove this theorem in the next section. Let us show now that Theorem 3.4 implies Theorem 3.2 (and so also Theorem A).

*Proof of Theorem 3.2.* Let  $X$  be a smooth curve over  $\mathbb{C}$ ,  $\tilde{X} \subset T^*X$  a smooth spectral curve. To simplify the notation, we will assume that  $p_{\tilde{X}} : \tilde{X} \rightarrow X$  is

ramified at a single point,  $\tilde{x} \in \tilde{X}$ . Denote by  $\tilde{X}^\wedge$  the formal completion of  $\tilde{X}$  at  $\tilde{x}$  and by  $X^\wedge$  the formal completion of  $X$  at  $x = p_{\tilde{X}}(\tilde{x})$ . Clearly,  $X^\wedge$  is a formal disc and  $\tilde{X}^\wedge$  is a (smooth ramified) spectral curve over  $X^\wedge$ .

It is a standard fact that the natural diagram

$$\begin{array}{ccc} \mathcal{Higgs}(\tilde{X}) & \rightarrow & \mathcal{Higgs}(\tilde{X}_u) \\ \downarrow & & \downarrow \\ \mathcal{Higgs}(\tilde{X}^\wedge) & \rightarrow & \mathcal{Higgs}(\tilde{X}_u^\wedge) \end{array}$$

is Cartesian; essentially, the claim is that a Higgs bundle on  $\tilde{X}$  can be glued from a Higgs bundle on  $\tilde{X}_u$ , a Higgs bundle on  $\tilde{X}^\wedge$ , and an identification of their restrictions to the punctured disc  $\tilde{X}_u^\wedge := \tilde{X}_u \cap \tilde{X}^\wedge$ . The same statement holds for groupoids  $\widetilde{\mathcal{Higgs}}(\bullet)$ ,  $\widetilde{\mathcal{Conn}}_\lambda(\bullet)$ , and  $\widetilde{\mathcal{Conn}}_\lambda(\bullet)$ . Now Theorem 3.2 follows from Theorem 3.4 by diagram chasing: it suffices to check that the diagrams

$$\begin{array}{ccc} \mathcal{Conn}_\lambda(\tilde{X}) & \rightarrow & \mathcal{C}(\tilde{X}) \\ \downarrow & & \downarrow \\ \mathcal{Conn}_\lambda(\tilde{X}^\wedge) & \rightarrow & \mathcal{C}(\tilde{X}^\wedge) \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{\mathcal{Conn}}_\lambda(\tilde{X}) & \rightarrow & \widetilde{\mathcal{C}}(\tilde{X}) \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{Conn}}_\lambda(\tilde{X}^\wedge) & \rightarrow & \widetilde{\mathcal{C}}(\tilde{X}^\wedge) \end{array}$$

are Cartesian.  $\square$

3.3. Let us now describe the groupoids from Theorem 3.4 explicitly. Set  $X = \mathrm{Spf} \mathbb{C}[[z]]$ . The cotangent bundle to  $X$  equals  $\mathrm{Spf} \mathbb{C}[[\xi]][[z]]$ , where  $\xi$  is the vector field  $\frac{d}{dz}$  on  $X$ . A spectral curve  $\tilde{X}$  over  $X$  is given by one equation

$$(3.5) \quad \xi^2 - t(z)\xi + d(z) = 0, \quad (t(z), d(z) \in \mathbb{C}[[z]]).$$

We will only consider the case when  $\tilde{X}$  is ramified over  $X$ , because only this case is needed for Theorem 3.2 (besides, the unramified case is simply a ‘formal’ version of Theorem 3.1). Since we also want the spectral curve to be smooth,  $t(z)$  and  $d(z)$  must satisfy the following condition:

$$(3.6) \quad \text{The discriminant } t^2(z) - 4d(z) \in \mathbb{C}[[z]] \text{ has a simple zero at } z = 0.$$

**Notation.** We denote by  $\Omega_X$  the  $\mathbb{C}[[z]]$ -module of (continuous) differentials of  $\mathbb{C}[[z]]$ ; it is a free  $\mathbb{C}[[z]]$ -module generated by  $dz$ . To simplify the notation, we will write  $Ldz$  instead of  $L \otimes_{\mathbb{C}[[z]]} \Omega_X$  for a  $\mathbb{C}[[z]]$ -module  $L$ .

$\mathcal{Higgs}(\tilde{X})$  is the groupoid of pairs  $(L, \nabla)$ , where  $L$  is a rank 2 free  $\mathbb{C}[[z]]$ -module and  $\nabla : L \rightarrow Ldz$  is a  $\mathbb{C}[[z]]$ -linear map such that  $\mathrm{tr} \nabla = t(z)dz$ ,  $\det(\nabla) = d(z)(dz)^2$ . Here  $\Omega_X$  is the free  $\mathbb{C}[[z]]$ -module with generator  $dz$ . The groupoid  $\mathcal{Higgs}(\tilde{X}_u)$  is similar, except  $L$  is a two-dimensional vector space over  $\mathbb{C}((z))$ .

$\mathcal{Conn}_\lambda(\tilde{X})$  is the groupoid of pairs  $(L, \nabla)$ , where  $L$  is a rank 2 free  $\mathbb{C}[[z, \lambda]]$ -module and  $\nabla : L \rightarrow Ldz$  is a  $(\mathbb{C}[[\lambda]]$ -linear)  $\lambda$ -connection such that the map  $\nabla_0 : L/\lambda L \rightarrow (L/\lambda L)dz$  induced by  $\nabla$  satisfies  $\mathrm{tr} \nabla_0 = t(z)dz$ ,  $\det(\nabla_0) = d(z)(dz)^2$ . The groupoid  $\mathcal{Higgs}(\tilde{X}_u)$  is similar, except  $L$  is a rank 2 free module over  $\mathbb{C}((z))[[\lambda]]$ .

$\mathcal{C}(\tilde{X})$  is the groupoid of triples  $(L_u, \nabla, L_0)$ , where  $(L_u, \nabla) \in \mathcal{Conn}_\lambda(\tilde{X}_u)$ , and  $L_0$  is a  $\mathbb{C}[[z]]$ -lattice in the vector space  $L_u/\lambda L_u$  such that  $\nabla_0(L_0) \subset L_0 dz$ . Here the Higgs field  $\nabla_0 : L_u/\lambda L_u \rightarrow (L_u/\lambda L_u)dz$  is induced by  $\nabla$ .

$\widetilde{\mathcal{Higgs}}(\tilde{X})$  (resp.  $\widetilde{\mathcal{Higgs}}(\tilde{X}_u)$ ) is the groupoid of rank 1 free  $\mathbb{C}[[\tilde{z}]]$ -modules (resp. one-dimensional  $\mathbb{C}((\tilde{z}))$ -vector spaces), where  $\tilde{z}$  is a formal coordinate on  $\tilde{X} \simeq$

$\mathrm{Spf} \mathbb{C}[[\tilde{z}]]$ . For instance, we can set

$$(3.7) \quad \tilde{z} = \sqrt{t(z)^2 - 4d(z)}.$$

$\widetilde{\mathrm{Conn}}_\lambda(\tilde{X})$  is the groupoid of pairs  $(l, \delta)$ , where  $l$  is a rank 1 free  $\mathbb{C}[[\tilde{z}, \lambda]]$ -module, and  $\delta : l \rightarrow \tilde{z}^{-1}ld\tilde{z}$  is a  $(\mathbb{C}[[\lambda]]$ -linear)  $\lambda$ -connection such that  $\mathrm{res} \delta : l/\tilde{z}l \rightarrow (\tilde{z}^{-1}l/l)d\tilde{z}$  equals  $-\lambda/2$ , and the induced map  $\delta_0 : l/\lambda l \rightarrow \tilde{z}^{-1}(l/\lambda l)d\tilde{z}$  equals the natural 1-form  $\mu \in \Omega_{\tilde{X}}$ . Notice that if  $\tilde{z}$  is given by (3.7), then  $\mu = (-t(z) + \tilde{z})dz/2$ . Similarly,  $\widetilde{\mathrm{Conn}}_\lambda(\tilde{X}_u)$  is the groupoid of pairs  $(l, \delta)$ , where  $l$  is a rank 1 free  $\mathbb{C}((\tilde{z}))[[\lambda]]$ -module, and  $\delta : l \rightarrow ld\tilde{z}$  is a  $\lambda$ -connection such that the induced map  $\delta_0 : l/\lambda l \rightarrow (l/\lambda l)d\tilde{z}$  equals  $\mu$ .

Finally,  $\tilde{\mathcal{C}}(\tilde{X})$  is the groupoid of triples  $(l_u, \delta, l_0)$ , where  $(l_u, \delta) \in \widetilde{\mathrm{Conn}}_\lambda(\tilde{X}_u)$ , and  $l_0$  is a  $\mathbb{C}[[z]]$ -lattice in the vector space  $l_u/\lambda l_u$ .

Let us now describe the natural functors between these groupoids. The functor  $\mathrm{Conn}_\lambda(\tilde{X}) \rightarrow \mathcal{C}(\tilde{X})$  (induced by the diagram (3.3)) sends  $(L, \nabla) \in \mathrm{Conn}_\lambda(\tilde{X})$  to  $(L \otimes_{\mathbb{C}[[z]]} \mathbb{C}((z)), \nabla, L/\lambda L) \in \mathcal{C}(\tilde{X})$ . Similarly, the functor  $\widetilde{\mathrm{Conn}}_\lambda(\tilde{X}) \rightarrow \tilde{\mathcal{C}}(\tilde{X})$  (induced by the diagram (3.4)) sends  $(l, \delta) \in \widetilde{\mathrm{Conn}}_\lambda(\tilde{X})$  to  $(l \otimes_{\mathbb{C}[[z]]} \mathbb{C}((z)), \delta, l/\lambda l) \in \tilde{\mathcal{C}}(\tilde{X})$ .

Finally, let us describe the equivalence  $\tilde{\mathcal{C}}(\tilde{X}) \xrightarrow{\sim} \mathcal{C}(\tilde{X})$ . Given  $(l_u, \delta, l_0) \in \tilde{\mathcal{C}}(\tilde{X})$ , we can consider  $l_u$  as a  $\mathbb{C}((z))[[\lambda]]$ -module using the embedding  $\mathbb{C}[[z]] \hookrightarrow \mathbb{C}[[\tilde{z}]]$ . Similarly,  $l_0$  can be viewed as a  $\mathbb{C}[[z]]$ -lattice in the two-dimensional  $\mathbb{C}((z))$ -space  $l_u/\lambda l_u$ . We have  $\Omega_X \otimes \mathbb{C}((\tilde{z})) = \Omega_{\tilde{X}} \otimes \mathbb{C}((\tilde{z}))$ , therefore  $l_u dz = l_u d\tilde{z}$  and we can view  $\delta$  as a  $\lambda$ -connection on the  $\mathbb{C}((z))[[\lambda]]$ -module  $l_u$ . In this sense, the functor  $\tilde{\mathcal{C}}(\tilde{X}) \xrightarrow{\sim} \mathcal{C}(\tilde{X})$  is forgetful: it sends  $(l_u, \delta, l_0)$  to the same triple  $(l_u, \delta, l_0)$ , but considered over  $\mathbb{C}[[z]]$ .

We will also need the following property of the equivalence  $\tilde{\mathcal{C}}(\tilde{X}) \xrightarrow{\sim} \mathcal{C}(\tilde{X})$ . Take any  $(l_u, \delta, l_0) \in \tilde{\mathcal{C}}(\tilde{X})$ , and let  $(L_u, \nabla, L_0)$  be its image in  $\mathcal{C}(\tilde{X})$  (so that  $l_u$  and  $L_u$  are identified as  $\mathbb{C}[[z]]$ -modules). Set  $\tilde{L}_u := L_u \otimes \mathbb{C}((\tilde{z}))$ . We then obtain a natural isomorphism

$$\phi : \tilde{L}_u \xrightarrow{\sim} l_u \oplus \sigma^* l_u,$$

where  $\sigma$  is the non-trivial element of the Galois group  $\mathrm{Gal}(\mathbb{C}((\tilde{z}))/\mathbb{C}((z)))$ . Notice that  $\nabla$  induces a  $\lambda$ -connection  $\tilde{\nabla} : \tilde{L}_u \rightarrow \tilde{L}_u d\tilde{z}$ , and  $\delta$  induces a  $\lambda$ -connection  $\delta \oplus \sigma^* \delta$  on  $l_u \oplus \sigma^* l_u$ . Besides,  $\tilde{L}_0 := L_0 \otimes \mathbb{C}((\tilde{z}))$  is a  $\mathbb{C}[[\tilde{z}]]$ -lattice in  $\tilde{L}_u/\lambda \tilde{L}_u$ , and  $l_0 \oplus \sigma^* l_0$  is a  $\mathbb{C}[[\tilde{z}]]$ -lattice in  $(l_u \oplus \sigma^* l_u)/\lambda(l_u \oplus \sigma^* l_u)$ . The following lemma is easy to prove:

**Lemma 3.5.** (1)  $\phi$  agrees with the  $\lambda$ -connections:  $\phi \circ \tilde{\nabla} = (\delta \oplus \sigma^* \delta) \circ \phi$ .

(2) The isomorphism

$$\phi_0 : \tilde{L}_u/\lambda \tilde{L}_u \xrightarrow{\sim} (l_u \oplus \sigma^* l_u)/\lambda(l_u \oplus \sigma^* l_u)$$

(induced by  $\phi$ ) satisfies  $\phi_0(\tilde{L}_0) \subset l_0 \oplus \sigma^* l_0$ .

(3) The image  $\phi_0(\tilde{L}_0) \subset l_0 \oplus \sigma^* l_0$  is the kernel of the composition

$$l_0 \oplus \sigma^* l_0 \rightarrow l_0/\tilde{z}l_0 \oplus (\sigma^* l_0)/(\tilde{z}\sigma^* l_0) = (l_0/\tilde{z}l_0)^2 \rightarrow (l_0/\tilde{z}l_0),$$

where the last map is  $(x, y) \mapsto x + y$ . In particular, the quotient  $(l_0 \oplus \sigma^* l_0)/\phi_0(\tilde{L}_0)$  is one-dimensional.

□

4.  $\lambda$ -CONNECTIONS ON FORMAL DISC: THEOREM 3.4

Let us keep the notation of Section 3.3. So  $X = \mathrm{Spf} \mathbb{C}[[z]]$ ,  $\tilde{X}$  is given by (3.5), and  $t(z), d(z) \in \mathbb{C}[[z]]$  satisfy (3.6).

## 4.1. Proof of Theorem 3.4(1).

**Lemma 4.1.** *Suppose  $A_0(z) \in \mathfrak{gl}_2(\mathbb{C}[[z]])$  is such that  $t(z) = \mathrm{tr} A_0(z)$ ,  $d(z) = \det A_0(z)$ .*

- (1)  *$A_0(z)$  is a regular element of  $\mathfrak{gl}_2(\mathbb{C}[[z]])$ .*
- (2) *Suppose  $B(z) \in \mathfrak{gl}_2(\mathbb{C}((z)))$  satisfies  $[B(z), A_0(z)] = 0$  and*

$$\frac{dB(z)}{dz} \in \mathfrak{gl}_2(\mathbb{C}[[z]]) + [A_0(z), \mathfrak{gl}_2(\mathbb{C}((z)))].$$

*Then  $B(z) \in \mathfrak{gl}_2(\mathbb{C}[[z]])$ .*

*Proof.* The first statement of the lemma is almost obvious; let us prove the second one. Notice that  $\mathrm{tr} \frac{dB}{dz}, \mathrm{tr} \left( \frac{dB}{dz} A_0 \right) \in \mathbb{C}[[z]]$ . Since  $[B(z), A_0(z)] = 0$ , and  $A_0(z)$  is regular, we can write

$$B(z) = f(z) + g(z)A_0(z)$$

for some  $f, g \in \mathbb{C}((z))$ . We have

$$(4.1) \quad \mathrm{tr} \frac{dB}{dz} = 2 \frac{df}{dz} + \frac{dg}{dz} \mathrm{tr} A_0 + g \mathrm{tr} \frac{dA_0}{dz} = 2 \frac{df}{dz} + \frac{dg}{dz} t + g \frac{dt}{dz},$$

and

$$(4.2) \quad \begin{aligned} \mathrm{tr} \left( \frac{dB}{dz} A_0 \right) &= \frac{df}{dz} \mathrm{tr} A_0 + \frac{dg}{dz} \mathrm{tr} (A_0^2) + g \mathrm{tr} \left( \frac{dA_0}{dz} A_0 \right) \\ &= \frac{df}{dz} t + \frac{dg}{dz} (t^2 - 2d) + g \left( t \frac{dt}{dz} - d \right). \end{aligned}$$

Therefore,

$$(4.3) \quad \mathrm{tr} \left( \frac{dB}{dz} (2A_0 - t) \right) = \frac{dg}{dz} (t^2 - 4d) + \frac{1}{2} g \frac{d(t^2 - 4d)}{dz}.$$

Clearly, (4.3) belongs to  $\mathbb{C}[[z]]$ ; by looking at the leading term of the Laurent expansion of (4.3), it is easy to see that  $g \in \mathbb{C}[[z]]$ . Now (4.1) implies  $\frac{df}{dz} \in \mathbb{C}[[z]]$ , so  $f \in \mathbb{C}[[z]]$ . Finally,  $B = f + gA_0 \in \mathfrak{gl}_2(\mathbb{C}[[z]])$ .  $\square$

*Remark 4.2.* Let us reformulate Lemma 4.1(2). Let  $\mathfrak{c} := \ker(\mathrm{ad} A_0)$  and  $\mathfrak{c}^\vee := \mathrm{coker}(\mathrm{ad} A_0)$  be the centralizer and the ‘co-centralizer’ of  $A_0 \in \mathfrak{gl}_2(\mathbb{C}[[z]])$ , respectively. Then  $\mathfrak{c}$  and  $\mathfrak{c}^\vee$  are rank 2 free  $\mathbb{C}[[z]]$ -modules, and  $\mathfrak{c} \otimes \mathbb{C}((z))$  and  $\mathfrak{c}^\vee \otimes \mathbb{C}((z))$  are the centralizer and the co-centralizer of  $A_0$  in  $\mathfrak{gl}_2(\mathbb{C}((z)))$ , respectively. Let  $D$  be the composition

$$\mathfrak{c} \otimes \mathbb{C}((z)) \hookrightarrow \mathfrak{gl}_2(\mathbb{C}((z))) \xrightarrow{d} \mathfrak{gl}_2(\mathbb{C}((z))) dz \rightarrow \mathfrak{c}^\vee \otimes \mathbb{C}((z)) dz.$$

Then Lemma 4.1(2) claims that  $D^{-1}(\mathfrak{c}^\vee dz) = \mathfrak{c}$ .

Note also that the composition  $\mathfrak{c} \rightarrow \mathfrak{gl}_2(\mathbb{C}[[z]]) \rightarrow \mathfrak{c}^\vee$  identifies  $\mathfrak{c}$  with a submodule of  $\mathfrak{c}^\vee$ . In this manner,  $D$  can be viewed as a connection (with a pole at  $z = 0$ ) on the  $\mathbb{C}[[z]]$ -module  $\mathfrak{c}$ . The monodromy of  $D$  has eigenvalues 1 and  $-1$ ; this follows from Theorem 11.6 of [2] (actually, Step 2 in the proof of [2, Proposition 12.5] suffices). It is not hard to derive Lemma 4.1(2) from this observation.

**Lemma 4.3.** *Suppose  $A(z, \lambda) \in \mathfrak{gl}_2(\mathbb{C}[[z, \lambda]])$  is such that  $\text{tr}(A(z, 0)) = t(z)$ ,  $\det(A(z, 0)) = d(z)$ , and  $R(z, \lambda) \in \text{GL}_2(\mathbb{C}((z))[[\lambda]])$  satisfies  $R(z, 0) \in \text{GL}_2(\mathbb{C}[[z]])$ . Set*

$$(4.4) \quad \text{Gauge}_\lambda(A, R) := R^{-1}AR + \lambda R^{-1} \frac{dR}{dz} \in \mathfrak{gl}_2(\mathbb{C}((z))[[\lambda]]),$$

*and suppose  $\text{Gauge}_\lambda(A, R) \in \mathfrak{gl}_2(\mathbb{C}[[z, \lambda]])$ . Then  $R(z, \lambda) \in \text{GL}_2(\mathbb{C}[[z, \lambda]])$ .*

*Remark 4.4.*  $\text{Gauge}_\lambda(A, R)$  is a  $\lambda$ -version of gauge transform: we can rewrite (4.4) as

$$\left( \lambda \frac{d}{dz} + \text{Gauge}_\lambda(A, R) \right) = R^{-1}(z, \lambda) \left( \lambda \frac{d}{dz} + A(z, \lambda) \right) R(z, \lambda).$$

*Proof.* Assume the converse. Let us expand

$$A(z, \lambda) = \sum_{i \geq 0} A_i(z) \lambda^i, \quad \text{Gauge}_\lambda(A, R) = \sum_{i \geq 0} A'_i(z) \lambda^i, \quad R(z, \lambda) = \sum_{i \geq 0} R_i(z) \lambda^i.$$

Note that  $A_i(z), A'_i(z) \in \mathfrak{gl}_2(\mathbb{C}[[z]])$  for  $i \geq 0$ ,  $R_0(z) \in \text{GL}_2(\mathbb{C}[[z]])$ , and  $R_i(z) \in \mathfrak{gl}_2(\mathbb{C}((z)))$  for  $i > 0$ .

Let  $j > 0$  be the minimal index such that  $R_j(z) \notin \mathfrak{gl}_2(\mathbb{C}[[z]])$ . Set

$$R' := \sum_{i=0}^{j-1} R_i(z) \lambda^i \in \text{GL}_2(\mathbb{C}[[z, \lambda]]), \quad B := \text{Gauge}_\lambda(A, R'), \quad Q := (R')^{-1}R.$$

Then  $\text{Gauge}_\lambda(A, R) = \text{Gauge}_\lambda(B, Q)$ . Replacing  $A$  with  $B$  and  $R$  with  $Q$ , we can assume without loss of generality that  $R' = 1$ , that is,  $R_0(z) = 1$  and  $R_i(z) = 0$  for  $0 < i < j$ .

Taking the coefficient of  $\lambda^j$  in (4.4), we now obtain

$$A'_j(z) = A_j(z) + [A_0(z), R_j(z)].$$

Therefore,  $[A_0(z), R_j(z)] \in \mathfrak{gl}_2(\mathbb{C}[[z]])$ . By Lemma 4.1(1), we know that  $A_0(z) \in \mathfrak{gl}_2(\mathbb{C}[[z]])$  is a regular element, and so we can write  $R_j(z) = S(z) + T(z)$  for some  $S(z) \in \mathfrak{gl}_2(\mathbb{C}[[z]])$ ,  $T(z) \in \mathfrak{gl}_2(\mathbb{C}((z)))$  such that  $[T(z), A_0(z)] = 0$ . Replacing  $A(z, \lambda)$  with  $\text{Gauge}_\lambda(A, 1 + S(z)\lambda^j)$  and  $R(z, \lambda)$  with  $(1 + S(z)\lambda^j)^{-1}R$ , we can assume  $S(z) = 0$ , so that  $[A_0(z), R_j(z)] = 0$ .

Now taking the coefficient of  $\lambda^{j+1}$  in (4.4), we obtain

$$A'_{j+1}(z) = A_{j+1}(z) + \frac{dR_j}{dz}(z) + [A_0(z), R_{j+1}(z)] + [A_1(z), R_j(z)] - [R_1(z)A_0(z), R_j(z)]$$

(the last term is non-zero only if  $j = 1$ ). Recall that  $[A_0(z), R_j(z)] = 0$  and  $A_0(z)$  is a regular element; therefore,

$$[A_1(z), R_j(z)] - [R_1(z)A_0(z), R_j(z)] \in [A_0(z), \mathfrak{gl}_2(\mathbb{C}((z)))].$$

Lemma 4.1(2) implies  $R_j(z) \in \mathfrak{gl}_2(\mathbb{C}[[z]])$ , which contradicts our assumption.  $\square$

Lemma 4.3 can be reformulated in terms of groupoid  $\mathcal{C}(\tilde{X})$ :

**Corollary 4.5.** *For any  $(L_u, \nabla, L_0) \in \mathcal{C}(\tilde{X})$ , there exists at most one  $\mathbb{C}[[z, \lambda]]$ -lattice  $L \subset L_u$  such that  $L_0 = L/\lambda L \subset L_u/\lambda L_u$  and  $\nabla(L) \subset Ldz$ .*

*Proof.* Suppose  $L \subset L_u$  is such a lattice. Choose a trivialization  $\iota : (\mathbb{C}[[z, \lambda]])^2 \xrightarrow{\sim} L$ , and define the *connection matrix* of  $\nabla$  to be  $A \in \mathfrak{gl}_2(\mathbb{C}[[z, \lambda]])$  such that

$$\left( \lambda \frac{d}{dz} + A(z, \lambda) \right) dz = \iota^{-1} \nabla \iota.$$

Now let  $L' \subset L_u$  be another such lattice; denote by  $A' \in \mathfrak{gl}_2(\mathbb{C}[[z, \lambda]])$  the connection matrix corresponding to some trivialization  $\iota' : (\mathbb{C}[[z, \lambda]])^2 \xrightarrow{\sim} L'$ . Clearly,  $A'$  is obtained from  $A$  by the ' $\lambda$ -gauge transform' (4.4):  $A' = \text{Gauge}_\lambda(A, R)$  for  $R(z, \lambda) := \iota^{-1}\iota' \in \text{GL}_2(\mathbb{C}((z))[[\lambda]])$ . As  $L/\lambda L = L'/\lambda L' \subset L_u/\lambda L_u$ , we see that  $R(z, 0) \in \text{GL}_2(\mathbb{C}[[z]])$ , so that the assumptions of Lemma 4.3 are satisfied. Therefore,  $R(z, \lambda) \in \text{GL}_2(\mathbb{C}[[z, \lambda]])$ , and  $L = L'$ .  $\square$

Recall that for a functor  $\mathcal{F} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$  between groupoids, the (*essential*) *fiber* of  $\mathcal{F}$  over  $\gamma_2 \in \mathcal{G}_2$  is the groupoid of pairs

$$\{(\gamma_1, f) \mid \gamma_1 \in \mathcal{G}_1, f : \mathcal{F}(\gamma_1) \xrightarrow{\sim} \gamma_2\}.$$

Given  $(L_u, \nabla, L_0) \in \mathcal{C}(\tilde{X})$ , the set of all  $\mathbb{C}[[z, \lambda]]$ -lattices  $L$  as in Corollary 4.5 is equivalent to the fiber of the functor  $\text{Conn}_\lambda(\tilde{X}) \rightarrow \mathcal{C}(\tilde{X})$ ; Corollary 4.5 claims the fiber is either empty or equivalent to a one-element set. Now Theorem 3.4(1) is implied by the following simple lemma:

**Lemma 4.6.** *Let  $\mathcal{F} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$  be a functor between groupoids.*

- (1)  *$\mathcal{F}$  is faithful if and only if the fiber of  $\mathcal{F}$  over any object of  $\mathcal{G}_2$  is discrete.*
- (2)  *$\mathcal{F}$  is fully faithful if and only if the fiber of  $\mathcal{F}$  over any object of  $\mathcal{G}_2$  is either empty or equivalent to a one-element set.*

$\square$

**4.2. Proof of Theorem 3.4(2).** The second statement of Theorem 3.4 is proved similarly to its first statement. Actually, Theorem 3.4(2) is simpler, because it deals with 'abelian' objects (line bundles); for instance, the  $\lambda$ -gauge transformation (4.4) simplifies. Finally, notice that  $\lambda$ -connections on line bundles can be reduced to ordinary connections (as in Remark 2.7), so all results of this section are more or less classical.

For any  $(l_u, \delta, l_0) \in \tilde{\mathcal{C}}(\tilde{X})$ , choose a trivialization  $\iota : \mathbb{C}((\tilde{z}))[[\lambda]] \xrightarrow{\sim} l_u$  such that the induced map  $\mathbb{C}((\tilde{z})) \xrightarrow{\sim} l_u/\lambda l_u$  identifies  $\mathbb{C}[[\tilde{z}]]$  and  $l_0$ . We will say that  $\iota$  respects  $l_0$ ; clearly, such  $\iota$  always exists. Denote by  $a(\tilde{z}, \lambda) \in \mathbb{C}((\tilde{z}))[[\lambda]]$  the connection matrix of  $\delta$  with respect to  $\iota$ :

$$\left( \lambda \frac{d}{d\tilde{z}} + a(\tilde{z}, \lambda) \right) d\tilde{z} = \iota^{-1} \delta \iota.$$

Recall now that the map  $\delta_0 : l/\lambda l \rightarrow (l/\lambda l)d\tilde{z}$  induced by  $\delta$  equals  $\mu$  (where  $\mu \in \Omega_{\tilde{X}}$  is the canonical 1-form on  $\tilde{X}$ ). Therefore,  $a(\tilde{z}, 0) = \mu(d\tilde{z})^{-1}$ .

**Lemma 4.7.** *Let  $(l_u, \delta, l_0)$ ,  $\iota$ , and  $a(\tilde{z}, \lambda)$  be as above. Denote by  $Y$  the set of formal series  $r(\tilde{z}, \lambda) \in \mathbb{C}((\tilde{z}))[[\lambda]]$  that satisfy the following conditions:*

$$(4.5) \quad r(\tilde{z}, 0) \in \mathbb{C}[[\tilde{z}]]^\times$$

$$(4.6) \quad \text{Set } a' := a + \lambda r^{-1} \frac{dr}{d\tilde{z}}. \text{ Then } a' \in \tilde{z}^{-1} \mathbb{C}[[\tilde{z}, \lambda]], \text{ and } \text{res}_{\tilde{z}=0} a' = -\lambda/2.$$

*Let  $\mathbb{C}[[z, \lambda]]^\times$  be the group of invertible Taylor power series of two variables; we let  $\mathbb{C}[[z, \lambda]]^\times$  act on  $Y$  by multiplication. Then the fiber of the functor  $\widetilde{\text{Conn}}_\lambda(\tilde{X}) \rightarrow \tilde{\mathcal{C}}(\tilde{X})$  over  $(l_u, \delta, l_0)$  is equivalent to the quotient set  $Y/\mathbb{C}[[z, \lambda]]^\times$ .*

*Proof.* The fiber of the functor is equivalent to the set of all  $\mathbb{C}[[\tilde{z}, \lambda]]$ -lattices  $l \subset l_u$  such that

$$(4.7) \quad l/\lambda l = l_0 \subset l_u/\lambda l_u$$

and

$$(4.8) \quad (l, \delta) \in \widetilde{\text{Conn}}_\lambda(\tilde{X}).$$

Any such lattice  $l$  can be written as

$$l = r\iota(\mathbb{C}[[\tilde{z}, \lambda]])$$

for some  $r(\tilde{z}, \lambda) \in (\mathbb{C}((\tilde{z}))[[\lambda]])^\times$ . Notice that  $r$  is unique up to multiplication by an invertible element of  $\mathbb{C}[[\tilde{z}, \lambda]]$ . Also,  $a'(\tilde{z}, \lambda) = \text{Gauge}_\lambda(a, r)$  is the connection matrix of  $\delta$  with respect to the trivialization  $r\iota : \mathbb{C}[[\tilde{z}, \lambda]] \xrightarrow{\sim} l$ . Now it is clear that (4.5) and (4.6) are equivalent to (4.7) and (4.8), respectively. This completes the proof.  $\square$

**Lemma 4.8.** *Suppose  $a(\tilde{z}, \lambda) \in \mathbb{C}((\tilde{z}))[[\lambda]]$  satisfies  $a(\tilde{z}, 0) = \mu(d\tilde{z})^{-1}$ . Define the set  $Y$  as in Lemma 4.7. Then either  $Y$  is empty or it consists of a single  $\mathbb{C}[[\tilde{z}, \lambda]]^\times$ -orbit.  $Y$  is non-empty if and only if  $a(\tilde{z}, \lambda)$  satisfies the following condition:*

$$(4.9) \quad a_1(\tilde{z}) \in \tilde{z}^{-1}\mathbb{C}[[\tilde{z}]] \text{ and } \operatorname{res}_{\tilde{z}=0} a(\tilde{z}, \lambda) = -\lambda/2.$$

Here  $a_1(\tilde{z}) := \frac{\partial a(\tilde{z}, \lambda)}{\partial \lambda}$  is the coefficient of  $\lambda$  in  $a(\tilde{z}, \lambda)$ .

*Proof.* Denote by  $Z$  the set of all series  $r(\tilde{z}, \lambda) \in \mathbb{C}((\tilde{z}))[[\lambda]]$  that satisfy (4.5), and consider the map

$$d \log : Z \rightarrow \mathbb{C}((\tilde{z}))[[\lambda]] : r(\tilde{z}, \lambda) \mapsto r(\tilde{z}, \lambda)^{-1} \frac{dr(\tilde{z}, \lambda)}{d\tilde{z}}.$$

Notice that  $Z$  is a group under multiplication and  $d \log$  is a group homomorphism. To prove the lemma, we need to verify two properties of  $d \log$ :

$$\begin{aligned} d \log(Z) &= \{s(\tilde{z}, \lambda) \in \mathbb{C}((\tilde{z}))[[\lambda]] \mid s(\tilde{z}, 0) \in \mathbb{C}[[\tilde{z}]], \operatorname{res}_{\tilde{z}=0} s(\tilde{z}, 0) = 0\} \\ (d \log)^{-1}(\mathbb{C}[[\tilde{z}, \lambda]]) &= \mathbb{C}[[\tilde{z}, \lambda]] \subset Z. \end{aligned}$$

Both properties are almost obvious, especially if one notices that for  $r(\tilde{z}, \lambda) \in Z$ , the expression  $\log(r) \in \mathbb{C}((\tilde{z}))[[\lambda]]$  makes sense.  $\square$

**Corollary 4.9.** *Let  $(l_u, \delta, l_0)$ ,  $\iota$ , and  $a(\tilde{z}, \lambda)$  be as in Lemma 4.7, and let  $a_1(\tilde{z}) := \frac{\partial a(\tilde{z}, \lambda)}{\partial \lambda}$  be the coefficient of  $\lambda$  in  $a(\tilde{z}, \lambda)$ . Then the fiber of the functor  $\widetilde{\text{Conn}}_\lambda(\tilde{X}) \rightarrow \tilde{\mathcal{C}}(\tilde{X})$  over  $(l_u, \delta, l_0)$  is either equivalent to a one-point set or empty. The fiber is non-empty if and only if (4.9) holds.*  $\square$

By Lemma 4.6, Corollary 4.9 is equivalent to Theorem 3.4(2). Let us also prove the following lemma, which is used in the next section.

**Lemma 4.10.** *All objects of  $\widetilde{\text{Conn}}_\lambda(\tilde{X})$  are isomorphic.*

*Proof.* Take any  $(l, \delta), (l', \delta') \in \text{Conn}_\lambda(\tilde{X})$ , and let us choose trivializations  $\iota : \mathbb{C}[[\tilde{z}, \lambda]] \xrightarrow{\sim} l$ ,  $\iota' : \mathbb{C}[[\tilde{z}, \lambda]] \xrightarrow{\sim} l'$ . Denote by  $a(\tilde{z}, \lambda), a'(\tilde{z}, \lambda)$  the connection matrices of  $\delta$  and  $\delta'$ , respectively. Notice that  $a - a' \in \lambda \mathbb{C}[[\tilde{z}, \lambda]]$ , because  $\operatorname{res}_{\tilde{z}=0} a = \operatorname{res}_{\tilde{z}=0} a'$  and  $a(\tilde{z}, 0) = a'(\tilde{z}, 0)$ . Therefore, there exists  $r \in (\mathbb{C}[[\tilde{z}, \lambda]])^\times$  such that

$$a'(\tilde{z}, \lambda) - a(\tilde{z}, \lambda) = \lambda r^{-1}(\tilde{z}, \lambda) \frac{dr(\tilde{z}, \lambda)}{d\tilde{z}} = \lambda d \log(r(\tilde{z}, \lambda)).$$

The map  $r(\tilde{z}, \lambda)\iota'\iota^{-1}$  provides an isomorphism  $(l, \delta) \xrightarrow{\sim} (l', \delta')$ .  $\square$

### 4.3. Proof of Theorem 3.4(3).

**Lemma 4.11.** *Suppose  $A(z, \lambda) = \sum_{i \geq 0} A_i(z) \lambda^i \in \mathfrak{gl}_2(\mathbb{C}((z))[[\lambda]])$  and  $R(z, \lambda) \in \mathrm{GL}_2(\mathbb{C}((z))[[\lambda]])$  satisfy the following conditions:*

- (1)  $A(z, \lambda)$  is diagonal;
- (2)  $R(z, 0) \in \mathfrak{gl}_2(\mathbb{C}[[z]])$ ;
- (3)  $\det R(z, 0) \in \mathbb{C}[[z]]$  has a first-order zero at  $z = 0$ ;
- (4)  $\tilde{A} := \mathrm{Gauge}_\lambda(A, R) \in \mathfrak{gl}_2(\mathbb{C}[[z, \lambda]])$  (for the definition of  $\mathrm{Gauge}_\lambda(A, R)$ , see (4.4)).

Then  $A_1(z) \in z^{-1} \mathfrak{gl}_2(\mathbb{C}[[z]])$ .

*Proof.* Set  $S(z, \lambda) = R(z, \lambda)R(z, 0)^{-1}$ ,  $B(z, \lambda) = \mathrm{Gauge}_\lambda(A, S)$ . Then  $B(z, \lambda) = \mathrm{Gauge}_\lambda(\tilde{A}, R(z, 0))$ ; this clearly implies  $B(z, \lambda) \in z^{-1} \mathfrak{gl}_2(\mathbb{C}[[z, \lambda]])$ .

Let us expand

$$B(z, \lambda) = \sum_{i \geq 0} B_i(z) \lambda^i, \quad S(z, \lambda) = \sum_{i \geq 0} S_i(z) \lambda^i,$$

where  $B_i(z) \in z^{-1} \mathfrak{gl}_2(\mathbb{C}[[z]])$ ,  $S_i(z) \in \mathfrak{gl}_2(\mathbb{C}((z)))$  for  $i \geq 0$ , and  $S_0(z) = 1$ . Taking the coefficient of  $\lambda$  in the identity  $B = \mathrm{Gauge}_\lambda(A, S)$ , we obtain

$$B_1(z) = A_1(z) + [A_0(z), S_1(z)].$$

Note that the diagonal entries of  $[A_0(z), S_1(z)]$  vanish (because  $A_0(z)$  is diagonal), and therefore  $A_1(z)$  and  $B_1(z)$  have the same diagonal entries. Therefore,  $A_1(z) \in z^{-1} \mathbb{C}[[z]]$  (because  $A_1(z)$  is diagonal).  $\square$

**Lemma 4.12.** *Let  $A$ ,  $R$ , and  $\tilde{A}$  be as in Lemma 4.11. Then  $\mathrm{res}_{z=0} \mathrm{tr} A(z, \lambda) = -\lambda$ .*

*Proof.* It is easy to see that

$$\mathrm{tr}(\tilde{A}) = \mathrm{tr}(A) + \lambda \frac{d(\det R(z))}{dz} (\det R(z))^{-1}.$$

Notice that  $\det R(z) = z f(z, \lambda)$ , where  $f(z, \lambda) \in \mathbb{C}((z))[[\lambda]]$  and  $f(z, 0) \in \mathbb{C}[[z]]^\times$ . Therefore,  $\ln f(z, \lambda)$  is well defined, and we can write

$$\frac{d(\det R(z))}{dz} (\det R(z))^{-1} = z^{-1} + \frac{df}{dz} f^{-1} = z^{-1} + \frac{d \ln f}{dz}.$$

Hence  $\mathrm{res}_{z=0} \mathrm{tr} A(z, \lambda) = \mathrm{res}_{z=0} \mathrm{tr} \tilde{A}(z, \lambda) - \lambda = -\lambda$ .  $\square$

**Proposition 4.13.** *Let  $(L_u, \nabla, L_0) \in \mathcal{C}(\tilde{X})$  be the image of  $(L, \nabla) \in \mathrm{Conn}_\lambda(\tilde{X})$  (so  $L_u = L \otimes \mathbb{C}((z))$  and  $L_0 = L/\lambda L \subset L_u/\lambda L_u$ ), and let  $(l_u, \delta, l_0)$  be the corresponding object of  $\tilde{\mathcal{C}}(\tilde{X})$ . Then  $(l_u, \delta, l_0) \in \tilde{\mathcal{C}}(\tilde{X})$  is isomorphic to the image of an object of  $\widetilde{\mathrm{Conn}}_\lambda(\tilde{X})$  (that is,  $(l_u, \delta, l_0)$  belongs to the essential image of  $\widetilde{\mathrm{Conn}}_\lambda(\tilde{X}) \rightarrow \tilde{\mathcal{C}}(\tilde{X})$ ).*

*Proof.* Choose a trivialization  $\mathbb{C}((\tilde{z}))[[\lambda]] \xrightarrow{\sim} l_u$  that respects  $l_0$  (as in Section 4.2). Denote by  $a(\tilde{z}, \lambda) = \sum_{i \geq 0} a_i(\tilde{z}) \lambda^i \in \mathbb{C}((\tilde{z}))[[\lambda]]$  the matrix of  $\delta$  in this trivialization. According to Corollary 4.9, we need to verify (4.9) to prove the proposition. We will do this by using Lemma 3.5.

Let  $\sigma$  be the non-trivial element of the Galois group  $\mathrm{Gal}(\mathbb{C}((\tilde{z}))/\mathbb{C}((z)))$ . The trivialization  $\mathbb{C}((\tilde{z}))[[\lambda]] \xrightarrow{\sim} l_u$  induces a trivialization  $(\mathbb{C}((\tilde{z}))[[\lambda]])^2 \xrightarrow{\sim} l_u \oplus \sigma^* l_u$ ; let  $A(\tilde{z}, \lambda) = \sum_{i \geq 0} A_i(\tilde{z}) \lambda^i \in \mathfrak{gl}_2(\mathbb{C}((\tilde{z}))[[\lambda]])$  be the matrix of the connection  $\delta \oplus \sigma^* \delta$

with respect to this trivialization. Note that  $A(\tilde{z}, \lambda)$  is a diagonal matrix, and one of its entries equals  $a(\tilde{z}, \lambda)$ . Besides,

$$\operatorname{res}_{\tilde{z}=0} \operatorname{tr} A(\tilde{z}, \lambda) = \operatorname{res}_{\tilde{z}=0} \delta + \operatorname{res}_{\tilde{z}=0} \sigma^* \delta = 2 \operatorname{res}_{\tilde{z}=0} \delta = 2 \operatorname{res}_{\tilde{z}=0} a(\tilde{z}, \lambda).$$

Therefore, it suffices to verify that  $A_1(\tilde{z}) \in \tilde{z}^{-1} \mathfrak{gl}_2(\mathbb{C}[[\tilde{z}]])$  and  $\operatorname{res}_{\tilde{z}=0} \operatorname{tr} A(\tilde{z}, \lambda) = -\lambda$ .

Let us choose a trivialization  $\iota : (\mathbb{C}[[\tilde{z}, \lambda]])^2 \xrightarrow{\sim} \tilde{L} := L \otimes_{\mathbb{C}[[z]]} \mathbb{C}[[\tilde{z}]]$ , and let  $\tilde{A}(\tilde{z}, \lambda) \in \mathfrak{gl}_2(\mathbb{C}[[\tilde{z}, \lambda]])$  be the matrix of the connection  $\tilde{\nabla} : \tilde{L} \rightarrow \tilde{L} d\tilde{z}$  (induced by  $\nabla$ ) with respect to  $\iota$ . From Lemma 3.5, we obtain a morphism  $\phi : \tilde{L} \rightarrow l \oplus l_u$  that respects the  $\lambda$ -connections. Therefore,  $\tilde{A} = \operatorname{Gauge}_\lambda(A, R)$ , where  $R$  is the matrix of  $\phi$ . Now the required properties of  $A(\tilde{z}, \lambda)$  follow from Lemmas 4.11 and 4.12.  $\square$

Now Theorem 3.4(3) easily follows. Indeed, Proposition 4.13 implies that under the isomorphism  $[\mathcal{C}(\tilde{X})] \xrightarrow{\sim} [\tilde{\mathcal{C}}(\tilde{X})]$ , the image of  $[\operatorname{Conn}_\lambda(\tilde{X})] \subset [\mathcal{C}(\tilde{X})]$  is contained in  $[\widetilde{\operatorname{Conn}_\lambda(\tilde{X})}] \subset [\mathcal{C}(\tilde{X})]$ . But  $[\widetilde{\operatorname{Conn}_\lambda(\tilde{X})}]$  is a one-element set (Lemma 4.10), and  $[\operatorname{Conn}_\lambda(\tilde{X})]$  is obviously not empty. Therefore, the isomorphism identifies the two sets.

## 5. PROOF OF THEOREM B

5.1. Let us start with a simple observation about foliations on formal schemes:

**Definition 5.1.** A  $\lambda$ -adic formal scheme is a formal scheme  $S$  together with a function  $\lambda \in H^0(S, \mathcal{O}_S)$  such that the zero locus of  $\lambda^{i+1}$  is a subscheme  $S_i \subset S$  and  $S = \varinjlim S_i$ . A  $\lambda$ -adic formal scheme  $S$  is *flat* if  $S_i$  is flat over  $\mathbb{C}[\lambda]/(\lambda^{i+1})$  for all  $i \geq 0$ , or, equivalently, if  $\lambda \in \mathcal{O}_S$  is not a zero divisor. Finally, a  $\lambda$ -adic formal scheme  $S$  is *smooth* if  $S_i$  is smooth over  $\mathbb{C}[\lambda]/(\lambda^{i+1})$  for all  $i \geq 0$ , or, equivalently, if  $S$  is flat and  $S_0$  is smooth over  $\mathbb{C}$ .

*Example 5.2.* For an arbitrary  $\mathbb{C}$ -scheme  $S$ , set  $S[[\lambda]] := \varinjlim S \times \operatorname{Spec} \mathbb{C}[\lambda]/(\lambda^i)$  (as in Definition 2.6). Then  $S[[\lambda]]$  is a flat  $\lambda$ -adic formal scheme; it is smooth if and only if  $S$  is smooth.

**Lemma 5.3.** *Let  $Y$  and  $Z$  be smooth  $\lambda$ -adic formal schemes, and  $\Phi : Y \rightarrow Z$  a morphism over  $\mathbb{C}[[\lambda]]$  (that is,  $\Phi^*(\lambda) = \lambda$ ). Denote by  $Y_0 \subset Y$  and  $Z_0 \subset Z$  the zero loci of  $\lambda$ .*

- (1) *If the restriction of  $\Phi$  to  $Y_0$  is smooth, then so is  $\Phi$ .*
- (2) *Suppose  $\Phi$  is smooth, and let  $\zeta := \ker(d\Phi) \subset TY$  be the foliation corresponding to the fibration  $\Phi : Y \rightarrow Z$ . Suppose that the quotient  $Y_0/\zeta$  exists and coincides with  $Z_0$  (that is, the restriction  $\Phi|_{Y_0} : Y_0 \rightarrow Z_0$  has connected non-empty fibers). Then the quotient  $Y/\zeta$  exists and coincides with  $Z$ .*

$\square$

Our proof of Theorem B is divided into the following steps:

- Construction of a map  $\Phi : \mathbf{M}_\#[[\lambda]] \rightarrow \mathbf{Conn}'_{\text{form}}$ .
- Verification that  $\Phi$  satisfies the assumptions of Lemma 5.3. Therefore,  $\mathbf{Conn}'_{\text{form}} = \mathbf{M}_\#[[\lambda]]/\zeta$ , where the foliation  $\zeta$  equals  $\ker(d\Phi)$ .
- Verification that  $\zeta = \zeta_\lambda$ .

**5.2. Construction of  $\Phi : \mathbf{M}_\#[[\lambda]] \rightarrow \mathbf{Conn}'_{form}$ .** Even though we formulated Theorems A, 3.2, and 3.4 for  $\mathbb{C}[[\lambda]]$ -families of  $\lambda$ -connections, the same proof works for  $K[[\lambda]]$ -families of  $\lambda$ -connections on a smooth curve  $X/K$ , where  $K$  is an arbitrary  $\mathbb{C}$ -algebra. The only change is that in Section 3.3, the modules over various algebras ( $K[[z, \lambda]]$ ,  $K((\tilde{z}))$ , and so on) have to be locally free, rather than free as before; accordingly, all calculations in Sections 4 have to be done locally on  $K$ .

*Remark 5.4.* Of course, the theorems also hold for families parametrized by  $S[[\lambda]]$  where  $S$  is a  $\mathbb{C}$ -scheme (or a stack); indeed, the statements are local on  $S$ . Even more generally, we can consider families parametrized by a flat  $\lambda$ -adic formal scheme. The details of this generalization are left to the reader. Notice that once Theorem B is proved, such statements become almost obvious.

Recall now that  $\mathbf{M}_\#$  is the moduli stack of triples  $(\tilde{X}, l, \partial)$ , where  $\tilde{X} \subset T^*X$  is a smooth spectral curve,  $l$  is a line bundle on  $\tilde{X}$ , and  $\partial : l \rightarrow l \otimes \Omega(\tilde{x}_1 + \cdots + \tilde{x}_n)$  is a connection whose residues at  $\tilde{x}_1, \dots, \tilde{x}_n$  (the ramification locus of  $p_{\tilde{X}} : \tilde{X} \rightarrow X$ ) equal  $-1/2$ .

Since  $\mathbf{M}_\#$  is a moduli stack, it carries a universal family; let us denote it by  $(\tilde{X}_\#, l_\#, \partial_\#)$ . Here  $\tilde{X}_\# \subset (T^*X) \times \mathbf{M}_\#$  is an  $\mathbf{M}_\#$ -family of smooth spectral curves,  $l_\#$  is a line bundle on  $\tilde{X}_\#$ , and  $\partial_\# : l_\# \rightarrow l_\# \otimes \Omega_{\tilde{X}_\#/\mathbf{M}_\#}(D_r)$  is a connections with pole at  $D_r$ , the ramification divisor of the projection  $\tilde{X}_\# \rightarrow X \times \mathbf{M}_\#$ . Let  $\mu = \mu_{\tilde{X}_\#} \in H^0(\tilde{X}_\#, \Omega_{\tilde{X}_\#/\mathbf{M}_\#})$  be the natural 1-form on  $\tilde{X}_\#$ ; it is the pull-back of the natural 1-form on  $T^*X$  under the projection  $\tilde{X}_\# \rightarrow T^*X$ .

Denote by  $l_\#[[\lambda]]$  the pull-back of  $l_\#$  to  $\tilde{X}_\#[[\lambda]]$ . The expression  $\mu + \lambda \partial_\#$  gives a  $\lambda$ -connection on  $l_\#[[\lambda]]$ :

$$\mu + \lambda \partial_\# : l_\#[[\lambda]] \rightarrow l_\#[[\lambda]] \otimes \Omega_{\tilde{X}_\#[[\lambda]]/\mathbf{M}_\#[[\lambda]]}(D_r[[\lambda]]).$$

So we see that over  $\mathbf{M}_\#[[\lambda]]$ , we have a natural family of spectral curves  $(\tilde{X}_\#[[\lambda]])$  and line bundles with  $\lambda$ -connections  $(l_\#[[\lambda]]$  and  $\mu + \lambda \partial_\#$ ) on these curves. According to the generalized Theorem 3.2, such a family corresponds to a  $\mathrm{GL}_2$ -bundle  $L$  on  $(\mathbf{M}_\# \times X)[[\lambda]]$  equipped with a  $\lambda$ -connection  $L \rightarrow L \otimes \Omega_X$ . In other words, we obtain a morphism  $\Phi : \mathbf{M}_\#[[\lambda]] \rightarrow \mathbf{Conn}_\lambda$  to the moduli stack of  $\mathrm{GL}_2$ -bundles with  $\lambda$ -connections on  $X$ . Clearly,  $\Phi(\mathbf{M}_\#[[\lambda]])$  is contained in  $\mathbf{Conn}'_{form}$  (the formal completion of  $\mathbf{Conn}_\lambda$  along  $\mathbf{Higgs}'$ ).

It is easy to see that Lemma 5.3(2) applies to  $\Phi : \mathbf{M}_\#[[\lambda]] \rightarrow \mathbf{Conn}_\lambda$ . Indeed,  $\mathbf{M}_\#$  is smooth, and the map

$$\lambda : \mathbf{Conn}_\lambda \rightarrow \mathbb{C} : (L, \nabla, \lambda) \mapsto \lambda$$

is smooth on  $\mathbf{Higgs}' \subset \mathbf{Conn}_\lambda$ ; therefore, both  $\mathbf{M}_\#[[\lambda]]$  and  $\mathbf{Conn}'_{form}$  are smooth  $\lambda$ -adic formal stacks. Besides, the restriction of  $\Phi$  to  $\mathbf{M}_\# \subset \mathbf{M}_\#[[\lambda]]$  (the zero locus of  $\lambda$ ) is the natural projection

$$\mathbf{M}_\# \rightarrow \mathbf{M}_\#/\zeta_0 = \mathbf{Higgs}'.$$

**5.3.** Now let us verify that the linear combination  $\zeta_\lambda = \zeta_0 - \lambda \zeta_\infty$  equals  $\zeta := \ker(d\Phi) \subset T(\mathbf{M}_\#[[\lambda]])$  (by Remark 2.12,  $\zeta_\lambda$  exists as a distribution). As  $\zeta_\lambda$  and  $\zeta$  have the same rank, it suffices to check  $\zeta_\lambda \subset \zeta$ . Equivalently, for any open set  $U \subset \mathbf{M}_\#[[\lambda]]$  and a vector field  $\theta$  on  $U$  that belongs to  $\zeta_\lambda$ , we need to check that  $\theta$  belongs to  $\zeta$ .

Set  $U[\epsilon] := U \times \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ . The vector field  $\theta$  induces an automorphism  $\Theta : U[\epsilon] \xrightarrow{\sim} U[\epsilon]$  characterized by the following property:

$$\Theta^*(f + \epsilon g) = f + \epsilon(g + \theta(f)) \quad (\text{here } f, g \in \mathcal{O}_U, \text{ so } f + \epsilon g \in \mathcal{O}_{U[\epsilon]}).$$

We need to verify that the two compositions

$$\Phi \circ \pi, \Phi \circ \pi \circ \Theta : U[\epsilon] \rightarrow \mathbf{Conn}_\lambda$$

coincide. Here  $\pi : U[\epsilon] \rightarrow U \hookrightarrow \mathbf{M}_\#[[\lambda]]$  is the natural projection.

Denote by  $(\tilde{X}_1, l_1, \delta_1)$  and  $(\tilde{X}_2, l_2, \delta_2)$  the pull-backs of the universal family  $(\tilde{X}_\#, l_\#, \mu + \lambda \partial_\#)$  under  $\pi : U[\epsilon] \rightarrow \mathbf{M}_\#[[\lambda]]$  and  $\pi \circ \Theta : U[\epsilon] \rightarrow \mathbf{M}_\#[[\lambda]]$ , respectively. Thus,  $\tilde{X}_i \subset TX \times U[\epsilon]$  is a  $U[\epsilon]$ -family of smooth spectral curves,  $l_i$  is a line bundle on  $\tilde{X}_i$ , and  $\delta_i$  is a  $\lambda$ -connection on  $l_i$  with the usual condition on the residues ( $i = 1, 2$ ). We need to verify that  $(\tilde{X}_1, l_1, \delta_1)$  and  $(\tilde{X}_2, l_2, \delta_2)$  define the same  $U[\epsilon]$ -family of  $\mathrm{GL}_2$ -bundles with  $\lambda$ -connections on  $X$ .

According to Theorem 3.2 (or rather its generalized version), this is equivalent to checking the following two statements:

- (1) The reductions of  $(\tilde{X}_1, l_1, \delta_1)$  and  $(\tilde{X}_2, l_2, \delta_2)$  modulo  $\lambda$  coincide.
- (2) Let  $\tilde{X}_{iu} \subset \tilde{X}_i$  be the open set where  $p_{\tilde{X}_i} : \tilde{X}_i \rightarrow U[\epsilon] \times X$  is unramified ( $i = 1, 2$ ). Then the push-forwards  $(p_{\tilde{X}_1})_*((l_1, \delta_1)|_{\tilde{X}_{1u}})$  and  $(p_{\tilde{X}_2})_*((l_2, \delta_2)|_{\tilde{X}_{2u}})$  are canonically isomorphic. Notice that the previous statement implies  $p_{\tilde{X}_1}(\tilde{X}_{1u}) = p_{\tilde{X}_2}(\tilde{X}_{2u})$ ; therefore, these push-forwards are  $\mathrm{GL}_2$ -bundles with  $\lambda$ -connections on the same open subset of  $X \times U[\epsilon]$ .

Both statements easily follow from the definition of  $\zeta_\lambda = \zeta_0 - \lambda \zeta_\infty$ .

This completes the proof of statements (2) and (3) of Theorem B. To prove Theorem B(1), we need to check that the fibration  $\zeta_\lambda \subset T\mathbf{M}_\#$  is a foliation. This translates into the vanishing of the curvature

$$\kappa : \zeta_\lambda \otimes \zeta_\lambda \rightarrow T\mathbf{M}_\#/\zeta_\lambda : \theta_1 \otimes \theta_2 \mapsto [\theta_1, \theta_2].$$

However,  $\kappa$  depends on  $\lambda$  algebraically, and Theorem B(2) implies that  $\kappa$  vanishes when  $\lambda \in \mathbb{C}[[\lambda]]$  is a formal parameter. The statement follows.

*Remark 5.5.* Theorem B(1) can be easily proved independently of Theorem B(2). By definition,  $\zeta_\lambda$  is a foliation if  $\lambda = 0$ , so we can assume  $\lambda \neq 0$ . Then  $\zeta_\lambda$  can be described as an isomonodromic deformation (similarly to  $\zeta_\infty$ ); the only difference is that the isomonodromic deformation uses twisted differential operators (where the twist depends on  $\lambda$ ).

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